

Trilinear generally covariant equations of AP

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Field equations for n -frames h^a_μ that are possible in the theory of absolute parallelism (AP) are considered. The methods of compatibility (or formal integrability) theory enable us to find the (non-lagrangian) equation having unusual kind of compatibility conditions, guaranteed by two (not one) identities. This 'unique equation' was not noted explicitly in the classification by Einstein and Mayer of compatible second order AP equations.

In this work it is shown that some equations of AP (including 'unique equation') can, after the substitution $h^a_\mu = H^p H^a_\mu$ ($H = \det H^a_\mu$, p is an n -dependent constant), be written in a trilinear form that contains only the matrix H^a_μ and its derivatives and not inverse matrix H^a_μ . The equations are still regular (and involutive) for degenerate but finite matrices H^a_μ if $\text{rank } H^a_\mu \geq 2$.

1. In this paper, we consider the theory of a Riemannian space with absolute parallelism proposed in [1-3]. A number of modern works (see [4,5] and references therein) deal with lagrangian equations (which are only a small part of Einstein–Mayer (EM) classification of compatible second order equations of AP [1]), but we want to show that it is not the most interesting of AP.

The geometry of AP is specified by a field of coframes

$$h^a_\mu \in GL(n),$$

which may be exposed (is determined up) to a global rotation:

$$h^{*a}_\mu(x) = s^a_b h^b_\mu(x), \text{ where } s^a_b = \text{const}; s^a_b \in O(1, n-1); \quad (1)$$

$$\text{i.e., } \eta_{ab} s^a_c s^b_d = \eta_{cd}, \text{ where } \eta_{ab} = \eta^{ab} = \text{diag}(-1, 1, \dots, 1) \text{ (Minkowski metric).}$$

With respect to coordinate transformations, the Latin and Greek indices have scalar and vector natures, respectively (for Lorenz transformations (1), all is quite the contrary). We introduce the metric

$$g_{\mu\nu} = \eta_{ab} h^a_\mu h^b_\nu = h^a_\mu h_{a\nu} \quad (2)$$

and the usual covariant differentiation with the symmetric connection. We also define the tensors (differential covariant)

$$\gamma_{a\mu\nu} = h_{a\mu;\nu}, \quad \Lambda_{a\mu\nu} = 2\gamma_{a[\mu\nu]} = h_{a\mu,\nu} - h_{a\nu,\mu}. \quad (3)$$

In covariant expressions, we shall omit in contractions the matrices

$$g^{\mu\nu} \text{ and } \eta^{ab} \text{ (since } g^{\mu\nu}_{;\lambda} = 0, \quad \eta^{ab}_{;\lambda} = 0),$$

understanding

$$\Psi_{\dots a \dots a \dots} = \Psi_{\dots a \dots b \dots} \eta^{ab}, \quad \Psi_{\dots \mu \dots \mu \dots} = \Psi_{\dots \mu \dots \nu \dots} g^{\mu\nu}. \quad (4)$$

The type of an index is changed by means of $h^a{}_\mu$; the notation (tensor identification) is unchanged (except for $h_{a\mu}$, $g_{\mu\nu}$, η_{ab}), for example:

$$\Lambda_{\mu\nu\lambda} = h_{a\mu} \Lambda_{a\nu\lambda}.$$

We introduce the notation ('scalar differentiation' and irreducible parts of Λ)

$$\Psi_{a\dots\mu,b} = \Psi_{a\dots\mu;\nu} h_{a\nu}. \quad (5)$$

$$\Phi_a = \Lambda_{bba}; \quad S_{abc} = 3\Lambda_{[abc]} = \Lambda_{abc} + (abc) = 6\gamma_{[abc]}; \quad (6)$$

$$f_{ab} = \Phi_{a,b} - \Phi_{b,a} + \Phi_c \Lambda_{cab} \quad \text{or} \quad f_{\mu\nu} = 2\Phi_{[\mu;\nu]} = \Phi_{\mu,\nu} - \Phi_{\nu,\mu}. \quad (7)$$

Note that

$$\Lambda_{abc} = -\Lambda_{acb}; \quad \gamma_{abc} = -\gamma_{bac} = 1/2 S_{abc} - \Lambda_{cab}. \quad (8)$$

For the transposition of scalar indices of differentiation there is a simple rule (here, Ψ is a scalar, i.e., has only Latin indices):

$$\Psi_{a\dots b,c,d} - \Psi_{a\dots b,d,c} = -\Psi_{a\dots b,e} \Lambda_{ecd}. \quad (9)$$

Sometimes, transition to Greek or mixed indices makes an expression more evident and plain; for example, from Eq. (3) there follows the well-known identity [1]:

$$\Lambda_{a[\mu\nu;\lambda]} \equiv 0 \quad \text{or} \quad \Lambda_{abc,d} + \Lambda_{ade} \Lambda_{ebc} + (bcd) \equiv 0. \quad (10)$$

Contracting two indices, we obtain [see (6)]:

$$\Lambda_{abc,a} + f_{bc} \equiv 0. \quad (11)$$

2. The equations of the frame field can be expressed in the form (separately symmetric and antisymmetric parts; see (3)–(7), [6]):

$$\begin{aligned} \mathbf{G}_{\mu\nu} &= 2\Lambda_{(\mu\nu)\lambda;\lambda} + \sigma(\Phi_{\mu;\nu} + \Phi_{\nu;\mu} - 2g_{\mu\nu}\Phi_{\lambda;\lambda}) + (\Lambda^2) \\ &= -2G_{\mu\nu} + (2\sigma - 2)(\Phi_{(\mu;\nu)} - g_{\mu\nu}\Phi_{\lambda;\lambda}) + U_{\mu\nu}(\Lambda^2) = 0, \end{aligned} \quad (12)$$

$$\mathbf{H}_{\mu\nu} = S_{\mu\nu\lambda;\lambda} + \tau f_{\mu\nu} + V_{\mu\nu}(\Lambda^2) = 0. \quad (13)$$

Here σ and τ are certain constants; $G_{\mu\nu}$ is Einstein tensor ($G_{\mu\nu;\nu} \equiv 0$); $U_{\mu\nu} = U_{\nu\mu}$, $V_{\mu\nu} = -V_{\nu\mu}$.

With regard to questions of compatibility (or formal integrability) of non-linear systems of partial differential equations, the reader is referred to [7]:

Corollary 4.11. The system \mathcal{R}_q is involutive (and compatible) if its symbol G_q is involutive and the mapping $\pi_q^{q+1} : \mathcal{R}_{q+1} \rightarrow \mathcal{R}_q$ is surjective.

If we differentiate the second-order system \mathcal{R}_2 [see Eqs. (12), (13)] and see that in a certain combination of equations (from system \mathcal{R}_3) the higher derivatives (h''') cancel, then if $\pi_2^3 : \mathcal{R}_3 \rightarrow \mathcal{R}_2$ is to be surjective the remaining terms must also cancel by means of \mathcal{R}_2 (otherwise a new second-order equation appears and it is, moreover, irregular in first jets), i.e., a corresponding identity must exist.

The symbol G_2 (vector space family over \mathcal{R}_2) of the system \mathcal{R}_2 can be determined by the linear system

$$G_2 : \quad \mathbf{e}_{ab} = \frac{\partial(\mathbf{G}_{ab} + \mathbf{H}_{ab})}{\partial h_{c\mu, \nu\lambda}} u_{c\mu, \nu\lambda} = 0, \quad (u_{c\mu, \nu\lambda} = u_{c\mu, \lambda\nu}). \quad (14)$$

(In coordinates $u_{a\mu, \nu \dots \tau}$ indices following 'comma' (here 'comma' does not mean coordinate derivative) are symmetric with respect to transpositions: $u_{a\mu, \nu \dots \tau} = u_{a\mu, (\nu \dots \tau)}$).

The involutory property of the symbol G_2 must be verified over all points (x, h, h', h'') in \mathcal{R}_2 , but Eq. (14) contains only the matrix h^a_μ (and inverse matrix h_a^μ), which can be made equal to the unit matrix by coordinate transformation provided h^a_μ is non-degenerate. If $h^a_\mu = \delta^a_\mu$, then Eq. (14) is closely similar to linearized equations.

We need also *Definition 2.16* of [7]: the symbol G_q is involutive if

$$\dim G_{q+1} = \dim G_q^0 + \dim G_q^1 + \dots + \dim G_q^i + \dots + \dim G_q^{n-1} \quad (\dim G_q^n \equiv 0). \quad (15)$$

For analyzing symbols, it is more convenient to use the Euclidean signature and indices $a, \mu = 1, \dots, n$ (while $0 \leq i \leq n$).

The subspaces G_2^i are determined by addition to Eq. (14) of the following equations ($G_2^0 = G_2$):

$$u_{ab, cd} = 0, \quad \text{if } c \leq i \text{ or } d \leq i. \quad (16)$$

These equations are non-covariant (with respect to infinitesimal coordinate diffeomorphisms and Lorenz transformations), but involutory property of symbol is covariant one [7], of course.

We turn to Eqs. (12) and (13). The equation $\mathbf{G}_{\mu\nu; \nu} = 0$ gives $f_{\mu\nu; \nu} = J_\mu^{(1)}$, or in Latin indices (for the interest of the thing)

$$\mathbf{G}_{ab, b} = (\sigma - 1)[f_{ab, b} - \frac{1}{2}\Lambda_{abc}f_{bc} + f_{ab}\Phi_b - J_a^{(1)}(\Lambda'\Lambda, \Lambda^3)] = 0. \quad (17)$$

From the equation $\mathbf{H}_{\mu\nu; \nu} = 0$ there follows the analogous ("Maxwell") equation

$$\mathbf{H}_{\mu\nu; \nu} = \tau(f_{\mu\nu; \nu} - J_\mu^{(2)}) = 0. \quad (18)$$

It is clear that (if $J_\mu^{(1)} = J_\mu^{(2)} = J_\mu$) the combination

$$\tau\mathbf{G}_{\mu\nu; \nu} + (1 - \sigma)\mathbf{H}_{\mu\nu; \nu}$$

must become identity when Eqs. (12) and (13) are taken into account:

$$\tau\mathbf{G}_{ab, b} + (1 - \sigma)\mathbf{H}_{ab, b} \equiv A_{abc}(\Lambda)\mathbf{G}_{bc} + B_{abc}(\Lambda)\mathbf{H}_{bc}. \quad (19)$$

Here A and B are linear in Λ .

It can be shown that the symbol G_2 is involutive if $\tau \neq 0$ ($\forall \sigma$), and therefore the system will be compatible in the presence of identity (19).

For $\sigma = 1$ (only the symmetric part occurs in identity) we can have the equation of vacuum general relativity (VGR):

$$-\frac{1}{2}\mathbf{G}_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0,$$

and in this case skew-symmetric part (13) can be arbitrary (if $\tau \neq 0$).

We come now finally to the most interesting case (details in [6]): $\tau = 0$, $\sigma \neq 0$. Identity (19) contains only the antisymmetric part, and for it there is a unique possibility:

$$\mathbf{H}_{\mu\nu} = S_{\mu\nu\lambda;\lambda} = 0; \quad \mathbf{H}_{\mu\nu;\nu} \equiv 0. \quad (20)$$

The symmetric part cannot be chosen arbitrary; for one must ensure that the equation

$$J_{\mu;\mu} = 0 \quad (J_{\mu} = J_{\mu}^{(1)}, \text{ see Eq. (17)}) \quad (21)$$

becomes an identity.

The symbol G_2 of the system (12)–(13) for $\tau = 0$, $\sigma \neq 0$ is not involutive, but its prolongation symbol G_3 is, and for compatibility it is necessary (and sufficient) to ensure also the second identity¹.

It can be shown [6] that there exists a unique compatible system with $\tau = 0$ (and $\sigma \neq 0$), for which $\sigma = 1/3$ and

$$f_{\mu\nu;\nu} = (S_{\mu\nu\lambda}\Phi_{\lambda})_{;\nu}. \quad (22)$$

It is convenient to express this unique equation of AP (in some sense, antipode of VGR in the set of AP equations) in the form

$$\frac{1}{2}(\mathbf{G}_{a\mu} - \mathbf{H}_{a\mu}) = L_{a\mu\nu;\nu} - \frac{1}{3}(f_{a\mu} + L_{a\mu\nu}\Phi_{\nu}) = 0 \quad (\sigma = 1/3, \tau = 0), \quad (23)$$

where

$$L_{a\mu\nu} = -L_{a\nu\mu} = \Lambda_{a\mu\nu} - S_{a\mu\nu} - 1/3(h_{a\mu}\Phi_{\nu} - h_{a\nu}\Phi_{\mu}). \quad (24)$$

The trace equation $\mathbf{G}_{aa} = 0$ [see (12), (23), (24)] becomes irregular for $n = 4$:

$$\mathbf{G}_{aa} = \frac{2}{3}(4 - n)\Phi_{aa} + Q(\Lambda^2) = 0, \quad Q \neq 0. \quad (25)$$

Therefore, additional spatial dimension(s) is needed.

3. We consider one further AP system (it looks simple in Latin indices)

$$\frac{1}{2}(\mathbf{G}_{ab} + \mathbf{H}_{ab}) = \Lambda_{abc,c} + \Lambda_{acd}\Lambda_{cdb} = 0 \quad (\sigma = 1, \tau = 0), \quad (26)$$

that, like Eq. (23), admits solutions without "electromagnetic field", i.e., remains compatible on addition of the equation

$$f_{\mu\nu} = 0. \quad (27)$$

System (26) was considered in [3], together with (27).

In general case, the tensor $V_{\mu\nu}$ in Eq. (13) can contain three terms:

$$V_{\mu\nu} = a_1 S_{\mu\nu\lambda}\Phi_{\lambda} + a_2 \Lambda_{\lambda\mu\nu}\Phi_{\lambda} + a_3 (\Lambda_{\mu\varepsilon\tau}\Lambda_{\varepsilon\tau\nu} - \Lambda_{\nu\varepsilon\tau}\Lambda_{\varepsilon\tau\mu}). \quad (28)$$

If $a_2 = 0$ and $a_3 = 0$, the system remains compatible on the addition of Eq. (27). Indeed, although the irregular equation $J_{\mu} = 0$ follows from Eqs. (18) and (27), it becomes an identity when (13) and (27) are taken into account:

$$J_{\mu} = J_{\mu}^{(2)} \sim (S_{\mu\nu\lambda}\Phi_{\lambda})_{;\nu} = -\frac{1}{2}S_{\mu\nu\lambda}f_{\nu\lambda} + \tau f_{\mu\lambda}\Phi_{\lambda}. \quad (29)$$

¹It is very beautifully.

For system (26), the symbol G_2 is determined by the equation (see (14); $h^a{}_\mu = \delta^a_\mu$)

$$\mathbf{e}_{ab} = u_{ab,cc} - u_{ac,cb} = 0, \quad (30)$$

(it differs from the Maxwell equation $A_{b,cc} - A_{c,cb} = 0$ only in the "redundant" index a).

We give the values of $\dim G_3$ and $\dim G_2^i$ for system (26):

$$\dim G_2^i = n^2(n-i)(n-i+1)/2 - n^2 + n^2\delta_n^i + n\delta_{n-1}^i; \quad (31)$$

$$\dim G_3 = n^3(n+1)(n+2)/6 - n^3 + n. \quad (32)$$

In (31), the first term expresses the number of coordinates $u_{ab,cd}$ with allowance for Eq. (16); the second, the number of equations (30); and the third, the fact that for G_2^n all equations (30) become an identity in view of (16). Finally, the last term $n\delta_{n-1}^i$ is added, since for G_2^{n-1} n equations in (30) become an identity (n means the fixed value of the index; no summation over it):

$$\mathbf{e}_{an} = u_{an,nn} - u_{an,nn} \equiv 0. \quad (33)$$

Terms of Eq. (32) have the same order: coordinates, equations, and identities.

Substituting (31) and (32), we can readily verify that involutory condition (15) is satisfied.

The expressions (31) and (32) are also valid for the general case ($\tau \neq 0$), but for Eqs. (23) we must add in (31) the term $+ \delta_{n-2}^i$, since in this case the equation

$$\mathbf{e}_{nm} - \mathbf{e}_{mn} = 0 \quad (m = n - 1)$$

in Eqs. (14) for G_2^{n-2} becomes an identity [in view of (16)]:

$$\mathbf{e}_{nm} - \mathbf{e}_{mn} \sim S_{nmc,c} = 0; \quad (34)$$

here $c \neq n, m$ (we recall that S_{abc} is completely antisymmetric tensor), and therefore this equation is contained in (16).

We also give expressions for $\dim G_3^i$ and $\dim G_4$, which are valid both for (23) and for the general case:

$$\dim G_3^i = n^2 C_3^{n-i+2} - n^2(n-i + n(1 - \delta_n^i)); \quad (35)$$

$$\dim G_3 = n^2 C_4^{n+4} - n^3(n+1)/2 + n^2. \quad (36)$$

Substituting these equations in condition (15), we can prove that the symbol G_3 is involutive.

4. In this section, we show that after the substitution

$$h_a{}^\mu = H^{-p} H_a{}^\mu \quad (p = (1 + 1/\sigma - n)^{-1}; \quad H = \det H_a{}^\mu; \quad h = \det h_a{}^\mu) \quad (37)$$

systems (23) and (26) can be rewritten in a form that contains only the matrix $H_a{}^\mu$ (and its coordinate derivatives) but not $H_a{}^\mu (H_a{}^\mu H_a{}^\nu = \delta_a^\nu)$.

Let us begin with Eq. (26), where $\sigma = 0$ and hence $H_a{}^\mu = h_a{}^\mu$. We rewrite (1) in the form

$$\Lambda_{abc} = -h_{a\mu}(h_b{}^\mu{}_{,\nu} h_c{}^\nu - h_c{}^\mu{}_{,\nu} h_b{}^\nu). \quad (38)$$

Using (38), we obtain in place of (23) the "trilinear" (by analogy with Hirota's bilinear form) equation:

$$-h_a{}^\mu \Lambda_{abc,c} - \Lambda_{cd}^\mu \Lambda_{cdb} = (h_b{}^\mu{}_{,\nu} h_c{}^\nu - h_c{}^\mu{}_{,\nu} h_b{}^\nu)_{,\lambda} h_c{}^\lambda + h_c{}^\mu{}_{,\lambda} (h_c{}^\lambda{}_{,\nu} h_b{}^\nu - h_b{}^\lambda{}_{,\nu} h_c{}^\nu) = 0. \quad (39)$$

The next step is the 'unique system' (23), where $\sigma = 1/3$ and $p = 1/(4 - n)$. Substituting (37) in (38) we obtain

$$\Lambda_{abc} = -2H^{-p} \left(H_{a\mu} H_{[b}^{\nu} H_{c]}^{\mu}{}_{,\nu} + p\eta_{a[b} H_{c]}^{\nu} A_{,\nu} \right), \quad (40)$$

$$\Phi_c = H^{-p} \left(H_c^{\nu}{}_{,\nu} + \frac{p}{\sigma} A_{,\nu} H_c^{\nu} \right) \quad (\text{here } A = \ln H). \quad (41)$$

We also need the expression for f_a^{μ} :

$$f_{\epsilon\tau} = \Phi_{\epsilon,\tau} - \Phi_{\tau\epsilon} = H_c^{\lambda}{}_{,\lambda\tau} H_{c\epsilon} + H_c^{\lambda}{}_{,\lambda} H_{c\nu} H_b^{\nu}{}_{,\epsilon} H_{b\tau} - (\epsilon\tau). \quad (42)$$

Rewriting Eq. (23) as follows

$$H^{-p} \left[(-h L_a^{\mu\nu})_{,\nu} + \frac{h}{3} (f_a^{\mu} + L_a^{\mu\nu} \Phi_{\nu}) \right] = 0, \quad (h = H^{np+1}) \quad (43)$$

we can ultimately reduce it (using the previous calculations) to the trilinear form

$$\begin{aligned} & \left(H_a^{\mu}{}_{,\nu\lambda} H_b^{\nu} - H_b^{\mu}{}_{,\nu\lambda} H_a^{\nu} - \frac{2}{3} H_a^{\nu}{}_{,\nu\lambda} H_b^{\mu} \right) H_b^{\lambda} + \frac{1}{3} H_b^{\nu}{}_{,\nu\lambda} \left(H_a^{\mu} H_b^{\lambda} + H_a^{\mu} H_b^{\lambda} \right) + H_a^{\mu}{}_{,\nu} (H_b^{\nu} H_b^{\lambda})_{,\lambda} \\ & + H_b^{\mu}{}_{,\nu} (H_b^{\nu}{}_{,\lambda} H_a^{\lambda} - 2H_a^{\nu}{}_{,\lambda} H_b^{\lambda} - H_b^{\lambda}{}_{,\lambda} H_a^{\nu}) + \frac{2}{9} H_b^{\lambda}{}_{,\lambda} (H_b^{\nu}{}_{,\nu} H_a^{\mu} - H_a^{\nu}{}_{,\nu} H_b^{\mu}) = 0. \end{aligned} \quad (44)$$

5. Examining trilinear equations (39) and (44), one may wonder whether the regularity of these equations is maintained at points of \mathcal{R}_2 at which the matrix H_a^{μ} is degenerate but finite. If the symbol G_2 (or, in the case of 'unique equation' (39), the symbol G_3) remains involutive, i.e., the numbers $\dim G_2^i$ ($\dim G_3^i$) do not change, then the answer will be in the affirmative. Of course, we must not now multiply, for example, Eq. (44) by H_a^{ϵ} , since some components of such system disappear, and there are changes in $\dim G_2$, $\dim G_3$, and so on, when the matrix H_a^{μ} is degenerate, i.e.,

$$r = \text{rank} H_a^{\mu} < n.$$

In addition, for $r < n$ the order of the indices is important for the calculation of the values $\dim G_q^i$ (see [7]); it need not be changed if by choice of the coordinates we represent H_a^{μ} in the form

$$H_a^{\mu} = \text{diag}(0, \dots, 0, 1, \dots, 1) = \delta_{\underline{a}}^{\mu} \quad (= \delta_{\underline{b}}^{\mu} \delta_{\underline{a}}^{\underline{b}}). \quad (45)$$

We use indices $\underline{a}, \underline{b} = n - r + 1, \dots, n$. Taking into account (45), one can write down an equation that determines the symbol $G_2(r)$ for the trilinear system (39):

$$\mathbf{e}_{ab} = v_{ba, \underline{c}\underline{c}} - v_{\underline{c}\underline{a}, \underline{c}\underline{b}}; \quad (46)$$

we change notation, $u_{ab} \rightarrow v_{ba}$, to emphasize that now the 'working matrix' is H_a^{μ} (but not coframe matrix h^a_{μ}). Suppose $r = 1$; then we obtain

$$\mathbf{e}_{ab} = v_{ba, nn} - v_{na, nn} \delta_b^n = 0, \quad \mathbf{e}_{an} \equiv 0; \quad (47)$$

here, n is a fixed value of the index, and there is no summation over n .

It is clear that $\dim G_2(r = 1) > \dim G_2(r = n)$, and Eqs. (39) are irregular for $r = 1$. It is easy to show that this conclusion also holds for Eqs. (44). Therefore, it is not possible to construct a formal solution in the form of a series beginning with $H_a^{\mu}(x_{(0)}^{\nu})$ of $r = 1$.

We now take $r = 2$ and in place of (46) write (we recall $m = n - 1$; m is also fixed value of index)

$$\mathbf{e}_{ab} = v_{ba,nn} + v_{ba,mm} - v_{na,n\bar{b}} - v_{ma,m\bar{b}} = 0. \quad (48)$$

It can be shown that $\dim G_2^i(2) = \dim G_2^i(n)$: see Eq. (31) and note the identity of the last equation with the equation for $G_2^i(n)$.

It is somewhat more complicated to show for 'unique system' (44) that $\dim G_3^i(2) = \dim G_3^i(n)$, but it can be done.

The conclusion is that trilinear equations (44) and (39)² are regular (involutive) if $r \geq 2$. At a point at which $r < n$, there will in general be a singularity, since the scalars Λ_{abc} become infinite. It is true that, depending on σ , there will be restrictions on n . It is very interesting (and very important) that for 'unique equation' and $n = 5$ the minor of 'working matrix' (which is the frame density of some weight) is equal to coframe matrix:

$$\partial H^{-1} / \partial H_a^\mu = H^{-1} H_a^\mu = h_a^\mu; \quad (49)$$

perhaps, this equality makes impossible arising (contra)singularities in general solutions (i.e., solutions of general position) of 'unique equation' for space-time dimension equal to five, because, as is shown in [8], another sort of singularities (co-singularities with $\text{rank } h_\mu^a \leq n - 1$) is impossible (only) for the 'unique equation' (23).

Two difficult questions to AP (Pauli) about 'energy-momentum tensor' and about 'post-newtonian effects' are also considered in [8]. Some non-trivial topological issues of AP, topological classification of symmetrical solutions and calculation of topological quasi-charge groups for $n = 5$ may be found in [9].

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²It turns out that one-parameter class II_{221221} of EM-classification [1] comes via these two trilinear equations; all equations of this class (and that is all) have trilinear presentation. This 'trilinear class' has one 'common point' with two-parameter class II_{22112} , but it has no common points with two-parameter class I_{12} of lagrangian equations.